Supersymmetric method for constructing quasi-exactly and conditionally-exactly solvable potentials

1999 J. Phys. A: Math. Gen. 32 1291

(http://iopscience.iop.org/0305-4470/32/7/019)

J. Phys. A: Math. Gen. **32** (1999) 1291–1300. Printed in the UK PII: S0305-4470(99)97132-9

Supersymmetric method for constructing quasi-exactly and conditionally-exactly solvable potentials

V M Tkachuk†

Ivan Franko Lviv State University, Chair of Theoretical Physics, 12 Drahomanov Str., Lviv UA–290005, Ukraine

Received 27 August 1998

Abstract. Using supersymmetric quantum mechanics we develop a new method for constructing quasi-exactly solvable (QES) potentials with two known eigenstates. This method is extended for constructing conditionally-exactly solvable potentials (CES). The considered QES potentials at certain values of parameters become exactly solvable and can be treated as CES ones.

1. Introduction

Since the appearance of quantum mechanics there has been continual interest in models for which the corresponding Schrödinger equation is exactly solvable. With regards to solvability of the Schrödinger equation there are three interesting classes of the potentials.

The first class is the exactly solvable potentials allowing us to obtain in explicit form all energy levels and corresponding wavefunctions. The hydrogen atom and harmonic oscillator are the best known examples of this type.

The second class is the so-called quasi-exactly solvable (QES) potentials for which a finite number of eigenstates of the corresponding Hamiltonian can be found exactly in explicit form. The first examples of QES potentials were given in [1–4]. Subsequently, several methods for generating QES potentials were worked out and as a result many QES potentials were found [5–13] (see also the review book [14]). Three different methods that are based respectively on the polynomial ansatz for wavefunctions, the point canonical transformation, and the supersymmetric (SUSY) quantum mechanics are described in [12]. Recently, an anti-isospectral transformation called a duality transformation was introduced in [15]. This transformation relates the energy levels and wavefunctions of two QES potentials. In [16] a new QES potential was discovered using this anti-isospectral transformation.

The third class is the conditionally-exactly solvable (CES) potentials for which the eigenvalues problem for the corresponding Hamiltonian is exactly solvable only when the parameters of the potential obey certain conditions. Such a class of potentials was first considered in [17]. It is interesting to note that in [18] it was demonstrated that the equivalence of the condition required for the potential obtained in [17] to be a CES potential with the condition that this potential can be put in an explicitly supersymmetric form. Recently, new examples of CES potentials have been discovered [19–21].

A very useful algebraic tool for the investigation of the problem of exact solvability of the Schrödinger equation is the SUSY quantum mechanics (for a review of SUSY quantum

† E-mail address: tkachuk@ktf.franko.lviv.ua

mechanics see [22, 23]). For constructing QES potentials the SUSY method was used in [10–12]. The starting point of this method is some initial QES potential with $n+1$ known eigenstates. Then applying the technique of SUSY quantum mechanics one can calculate the supersymmetric partner of the QES potential which is a new QES potential with n known eigenstates. In [20, 21] the SUSY quantum mechanics was used to develop some generalized method for constructing the CES potentials.

In our previous paper [24] we have proposed a new SUSY method for constructing QES potentials with two known eigenstates which, in contrast to [10–12], does not require knowledge of the initial QES potentials for generation of new QES ones. In [25] we extended this method for constructing QES potentials with three known eigenstates.

This paper is devoted to the further development of the SUSY method proposed in [24] and to extend this method for the construction of CES potentials. We obtain new QES and CES potentials. An interesting new point is that QES potentials with two known eigenstates become exactly solvable at certain fixed values of parameter and can, therefore, be treated as CES potentials.

2. Supersymmetric quantum mechanics

In Witten's model of supersymmetric quantum mechanics the SUSY partner Hamiltonians $H_\pm$ read

$$H_\pm = B^\mp B^\pm = -\frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}x^2} + V_\pm(x) \tag{1}$$

where

$$B^\pm = \frac{1}{\sqrt{2}}\left(\mp\frac{\mathrm{d}}{\mathrm{d}x} + W(x)\right) \tag{2}$$

$$V_\pm(x) = \tfrac{1}{2}(W^2(x) \pm W'(x)) \qquad W'(x) = \frac{\mathrm{d}W(x)}{\mathrm{d}x} \tag{3}$$

$W(x)$ is referred to as a superpotential. In this paper we shall consider the systems on the full real line $-\infty < x < \infty$.

The eigenvalues $E_n^\pm$ and eigenfunctions $\psi_n^\pm(x)$ of the Hamiltonians $H_\pm$ are related by SUSY transformations which in the case of unbroken SUSY read

$$E_{n+1}^- = E_n^+ \qquad E_0^- = 0 \tag{4}$$

$$\psi_{n+1}^-(x) = \frac{1}{\sqrt{E_n^+}}B^+\psi_n^+(x) \qquad \psi_n^+(x) = \frac{1}{\sqrt{E_{n+1}^-}}B^-\psi_{n+1}^-(x). \tag{5}$$

As a consequence of SUSY the Hamiltonians H_+ and H_- have the same energy spectrum except for the zero-energy ground state. The latter exists in the case of the unbroken SUSY. Only one of the Hamiltonians $H_\pm$ has a square integrable eigenfunction corresponding to the zero-energy. Here we use the convention that the zero energy eigenstate belongs to H_-. Due to the factorization of the Hamiltonians $H_\pm$ (see (1)) the ground state for H_- satisfies the equation $B^-\psi_0^-(x) = 0$, the solution of which is

$$\psi_0^-(x) = C_0^- \exp\left(-\int W(x)\,\mathrm{d}x\right). \tag{6}$$

C_0^- is the normalization constant. Here and below, C denotes the normalization constant of the corresponding wavefunction. From the condition of square integrability of the wavefunction $\psi_0^-(x)$ it follows that the superpotential must satisfy the condition

$$\mathrm{sign}(W(\pm\infty)) = \pm 1 \tag{7}$$

which is the condition of the existence of unbroken SUSY. For a detailed description of SUSY quantum mechanics and its application for the exact calculation of eigenstates of Hamiltonians see, [22, 23]. The properties of the unbroken SUSY quantum mechanics which are reflected in SUSY transformations (4) and (5) can be used for an exact calculation of the energy spectrum and wavefunctions. In this paper we use these properties for the generation of the QES potentials with two known eigenstates and CES potentials.

3. QES potentials with two known eigenstates

We shall solve the eigenvalue problem for the Hamiltonian H_-. The ground state of this Hamiltonian is known and is given by wavefunction (6) with energy $E_0^- = 0$. In order to calculate the excited state of H_- we use the following well known procedure used in SUSY quantum mechanics. Let us consider the SUSY partner of H_-, i.e. the Hamiltonian H_+. If we calculate the ground state of H_+ we immediately find the first excited state of H_- using the SUSY transformations (4), (5). In order to calculate the ground state of H_+ let us rewrite it in the following form:

$$H_+ = H_-^{(1)} + \epsilon = B_1^+ B_1^- + \epsilon \qquad \epsilon > 0 \tag{8}$$

which leads to the following relation between the potential energies:

$$V_+(x) = V_-^{(1)}(x) + \epsilon \tag{9}$$

and superpotentials

$$W^2(x) + W'(x) = W_1^2(x) - W_1'(x) + 2\epsilon \tag{10}$$

where ϵ is the energy of the ground state of H_+ since we supposed that $H_-^{(1)}$ similarly to H_- has zero-energy ground state, $B_1^\pm$ and $V_-^{(1)}(x)$ are given by (2) and (3) with the new superpotential $W_1(x)$.

As we see from (8) the ground state wavefunction of H_+ is also the ground state wavefunction of $H_-^{(1)}$ and it satisfies the equation $B_1^- \psi_0^+(x) = 0$. The solution of this equation is

$$\psi_0^+(x) = C_0^+ \exp\left(- \int W_1(x)\,\mathrm{d}x \right) \tag{11}$$

where for the square integrability of this function the superpotential $W_1(x)$ must satisfy the same condition as $W(x)$ (7). Using (4) and (5) we obtain the energy level $E_1^- = \epsilon$ and the wavefunction of the first excited state $\psi_1^-(x)$ for H_-.

Repeating this procedure in the case of shape invariant potentials [26] and self-similar potentials [27, 28] it is possible to calculate all of the energy spectrum and the corresponding wavefunctions. As a result of these cases many exactly solvable potentials were obtained [29] (see also [22]).

We consider a more general case and do not restrict ourselves to the shape invariant potentials or self-similar potentials. In this case it is not possible to obtain all of the energy spectrum. In [24] we obtained the general solution of equation (10) and thus derived a general expression for the QES potential with two explicitly known eigenstates. The basic idea consists of finding such a pair of $W(x)$ and $W_1(x)$ that satisfies equation (10). To do this we rewrite equation (10) in the following form

$$W_+'(x) = W_-(x) W_+(x) + 2\epsilon \tag{12}$$

where

$$\begin{aligned} W_+(x) &= W_1(x) + W(x) \\ W_-(x) &= W_1(x) - W(x). \end{aligned} \tag{13}$$

This new equation (12) can be easily solved with respect to $W_-(x)$ for a given arbitrary function $W_+(x)$ or with respect to $W_+(x)$ for a given arbitrary function $W_-(x)$. Then from (13) we obtain superpotentials $W(x)$ and $W_1(x)$ which satisfy equation (10).

In contrast to our paper [24] where we use the solution with respect to $W_-(x)$, in this paper we use the solution with respect to $W_+(x)$. This solution is more convenient for the construction of CES potentials and can be written in the following form

$$W_+(x) = \exp\left(\int \mathrm{d}x\, W_-(x)\right)\left[2\epsilon \int \mathrm{d}x \exp\left(-\int \mathrm{d}x\, W_-(x)\right) + \lambda\right] \tag{14}$$

here λ is the constant of integration. In order to simplify solution (14) let us choose $W_-(x)$ to be of the form

$$W_-(x) = -\phi''(x)/\phi'(x). \tag{15}$$

To provide a nonsingularity of $W_-(x)$ and as a result a nonsingularity of $V_\pm(x)$ we shall consider a nonsingular monotonic function $\phi(x)$ satisfying the condition $\phi'(x) > 0$. Then, substituting (15) into (14), we obtain

$$W_+(x) = (2\epsilon\phi(x) + \lambda)/\phi'(x). \tag{16}$$

Note that the constant λ can be included into the function $\phi(x)$ and thus for $W_+(x)$ we obtain

$$W_+(x) = 2\epsilon\phi(x)/\phi'(x). \tag{17}$$

Finally, for superpotentials $W(x)$ and $W_1(x)$ we have

$$W(x) = (\epsilon\phi(x) + \tfrac{1}{2}\phi''(x))/\phi'(x) \tag{18}$$

$$W_1(x) = (\epsilon\phi(x) - \tfrac{1}{2}\phi''(x))/\phi'(x). \tag{19}$$

Using this result for the wavefunctions of the ground state with the energy $E_0^- = 0$ and excited state with $E_1^- = \epsilon$ we obtain

$$\psi_0^-(x) = C_0^-(\phi'(x))^{-1/2} \exp\left(-\epsilon \int \mathrm{d}x\, \phi(x)/\phi'(x)\right) \qquad E_0^- = 0 \tag{20}$$

$$\psi_1^-(x) = C_1^-\phi(x)(\phi'(x))^{-1/2} \exp\left(-\epsilon \int \mathrm{d}x\, \phi(x)/\phi'(x)\right) \qquad E_1^- = \epsilon. \tag{21}$$

Note that as we see from (13) $W_+(x)$ must satisfy the same condition (7) as $W(x)$ and $W_1(x)$ do. Then, because $\phi(x)$ is a monotonic function and $\phi'(x) > 0$ from (17), it follows that $\phi(x)$ has one node. Therefore, $\psi_1^-(x)$ given by (21) also has one node and thus corresponds to the first excited state. The functions $\phi(x)$ that satisfy the described condition also provide the square integrability of the wavefunctions (20) and (21).

It is worth stressing that $\phi(x) = \psi_1^-(x)/\psi_0^-(x)$ from which follows an interesting fact. Namely, the ratio $\phi(x)$ of the wavefunctions of the first excited state and ground state and the distance ϵ between the corresponding energy levels entirely determine the potential energy.

The QES potential $V_-(x)$ is given by (3) with superpotential (18). Choosing different $\phi(x)$ and ϵ we obtain different QES potentials with two explicitly known eigenstates.

Now let us consider some interesting examples of new QES potentials which become exactly solvable at certain fixed values of parameter ϵ and can thus be treated as CES ones.

3.1. Example 1

Let us put

$$\phi(x) = \beta H_{2k+1}(\mathrm{i}x) \tag{22}$$

where $H_m(x)$ is a Hermite polynomial. The final result does not depend on the constant β. The superpotentials in this case read

$$W(x) = \gamma x + \mathrm{i}2k(\gamma+1)\frac{H_{2k-1}(\mathrm{i}x)}{H_{2k}(\mathrm{i}x)} \tag{23}$$

$$W_1(x) = \gamma x + \mathrm{i}2k(\gamma-1)\frac{H_{2k-1}(\mathrm{i}x)}{H_{2k}(\mathrm{i}x)} \tag{24}$$

where we have introduced the notation

$$\gamma = \frac{\epsilon}{2k+1}. \tag{25}$$

Substituting the superpotential $W(x)$ (23) into (3) we obtain the following QES potential $V_-(x)$:

$$\begin{aligned} V_-(x) = {}& \frac{1}{2}\gamma^2 x^2 + 2k(2k-1)(\gamma+1)^2\frac{H_{2k-2}(\mathrm{i}x)}{H_{2k}(\mathrm{i}x)} \\ & -2k^2(\gamma+1)(\gamma+3)\left(\frac{H_{2k-1}(\mathrm{i}x)}{H_{2k}(\mathrm{i}x)}\right)^2 + k\gamma(\gamma+1) - \frac{1}{2}\gamma. \end{aligned} \tag{26}$$

The wavefunctions of the ground and first excited states read

$$\psi_0^-(x) = C_0^-(H_{2k}(\mathrm{i}x))^{-(1+\gamma)/2}\exp(-\gamma x^2/2) \tag{27}$$

$$\psi_1^-(x) = C_1^- H_{2k+1}(\mathrm{i}x)(H_{2k}(\mathrm{i}x))^{-(1+\gamma)/2}\exp(-\gamma x^2/2). \tag{28}$$

It is interesting to note that in the special case $\gamma = 1$, i.e.

$$\epsilon = 2k+1 \tag{29}$$

the second term in $W_1(x)$ falls out and $W_1(x)$ corresponds to the superpotential of a linear harmonic oscillator. Then $V_-^{(1)}(x)$ and, as a result of (9), $V_+(x)$ are the potential energies of the linear harmonic oscillator. Therefore, in this case, the SUSY partner H_+ is the Hamiltonian of the linear harmonic oscillator and we know all its eigenfunctions in explicit form. Using SUSY transformations (4), (5) we can easily calculate the energy levels and the wavefunctions of all the excited states of H_-. The energy spectrum of H_- in this special case is the following:

$$E_0^- = 0 \qquad E_n^- = n + 2k \qquad n = 1, 2, \ldots. \tag{30}$$

Thus the QES potential (26) at a fixed value of ϵ (29) becomes exactly solvable and can therefore be treated as the CES potential. Note that $V_-(x)$ in this special case $\gamma = 1$ corresponds to the potential obtained by Bagrov and Samsonov [30, 31] via the Darboux method and later by Junker and Roy [20, 21] within the SUSY approach.

3.2. *Example 2*

Consider the function

$$\phi(x) = \beta\frac{H_{2k+1}(\mathrm{i}x)}{H_{2m}(\mathrm{i}x)} \qquad k \geqslant m \tag{31}$$

which generalizes the one given in the first example. For superpotentials we obtain

$$W(x) = -x - \mathrm{i}4m\frac{H_{2m-1}(\mathrm{i}x)}{H_{2m}(\mathrm{i}x)} - \mathrm{i}\frac{\epsilon + 2k - 2m + 1}{H_{2m+1}(\mathrm{i}x)/H_{2m}(\mathrm{i}x) - H_{2k+2}(\mathrm{i}x)/H_{2k+1}(\mathrm{i}x)} \tag{32}$$

$$W_1(x) = x + \mathrm{i}4m\frac{H_{2m-1}(\mathrm{i}x)}{H_{2m}(\mathrm{i}x)} - \mathrm{i}\frac{\epsilon - (2k - 2m + 1)}{H_{2m+1}(\mathrm{i}x)/H_{2m}(\mathrm{i}x) - H_{2k+1}(\mathrm{i}x)/H_{2k}(\mathrm{i}x)}. \tag{33}$$

The QES potential $V_-(x)$ is given by (3) with superpotential (32). The expressions for the QES potential and the wavefunctions in this case are somewhat complicated and we do not write them down in explicit form.

We would like to stress the following very interesting point. In the special case

$$\epsilon = 2k - 2m + 1 \tag{34}$$

the second term in superpotential $W_1(x)$ (33) drops up and then this $W_1(x)$ coincides with superpotential $W(x)$ (23) from the first example for $\gamma = 1$ which is exactly solvable. Then, using the same explanation as in the end of the first example, we may conclude that for the potential $V_-(x)$ calculated with superpotential (32) in the special case (34) it is possible to obtain all energy levels and corresponding wavefunctions and thus $V_-(x)$ can be treated as the CES potential. For the energy levels we obtain

$$\begin{aligned} &E_0^- = 0 \qquad E_1^- = 2k - 2m + 1 \\ &E_n^- = n + 2k \qquad n = 2, 3, 4, \ldots . \end{aligned} \tag{35}$$

In this special case the potential energy $V_-(x)$ coincides with the one studied in [30, 31].

4. CES potentials

In this section we develop a consistent method for constructing the CES potentials using the results of the previous section.

Suppose that $W_1(x)$ is a given superpotential that corresponds to the exactly solvable potential $V_-^{(1)}(x)$. The example of such a superpotential is a shape invariant one [26]. As a result of (9) $V_+(x)$ is also exactly solvable and thus for H_+ we know in explicit form all energy levels and the corresponding eigenfunctions. Then using SUSY transformations (4), (5) we can easily calculate all excited energy levels and wavefunctions of its SUSY partner H_-, the wavefunction of the ground state is given by (6). But to do this we must have the superpotential $W(x)$ which is expressed through $\phi(x)$. Because $W_1(x)$ is a given function it is convenient to represent the superpotential $W(x)$ using (18) and (19) in the following form

$$W(x) = W_1(x) + \frac{\phi''(x)}{\phi'(x)}. \tag{36}$$

Similarly, the new exactly solvable potential $V_-(x)$ can be written as follows

$$V_-(x) = \frac{1}{2}(W_1^2(x) + W_1'(x)) + \left(\frac{\phi''(x)}{\phi'(x)}\right)^2 + 2W_1(x)\frac{\phi''(x)}{\phi'(x)} - \epsilon. \tag{37}$$

In this expression the function $\phi(x)$ is not an arbitrary one but must satisfy (19) for a given $W_1(x)$. Thus we must solve equation (19) with respect to $\phi(x)$ for a given $W_1(x)$ which can be written in the following form

$$\tfrac{1}{2}\phi''(x) + W_1(x)\phi'(x) = \epsilon\phi(x). \tag{38}$$

In order to transform this equation into a Schrödinger-type equation let us write $\phi(x)$ in the form

$$\phi(x) = f(x)\exp\left(-\int \mathrm{d}x\, W_1(x)\right). \tag{39}$$

The new function $f(x)$ satisfies the equation which can be rewritten as follows

$$-\tfrac{1}{2}f''(x) + V_+^{(1)}(x)f(x) = -\epsilon f(x) \tag{40}$$

where

$$V_+^{(1)}(x) = \tfrac{1}{2}(W_1^2(x) + W_1'(x)). \tag{41}$$

As we see it is a Schrödinger-type equation of SUSY quantum mechanics but with negative energy. The sign in the right-hand side of equation (40) can be changed using a duality transformation called an anti-isospectral transformation [15]:

$$\xi = \mathrm{i}x. \tag{42}$$

Then equation (40) reads

$$-\frac{1}{2}\frac{\mathrm{d}^2\tilde{f}(\xi)}{\mathrm{d}\xi^2} + \tilde{V}_-^{(1)}(\xi)\tilde{f}(\xi) = \epsilon\tilde{f}(\xi) \tag{43}$$

where we have introduced the notations

$$\tilde{f}(\xi) = f(-\mathrm{i}\xi) \tag{44}$$

$$\tilde{V}_-^{(1)}(\xi) = -V_+^{(1)}(-\mathrm{i}\xi) = \frac{1}{2}\left(\tilde{W}_1^2(\xi) - \frac{\mathrm{d}\tilde{W}_1(\xi)}{\mathrm{d}\xi}\right) \tag{45}$$

$$\tilde{W}_1(\xi) = \mathrm{i}W_1(-\mathrm{i}\xi). \tag{46}$$

In this paper we shall consider only such superpotentials $W_1(x)$ for which the dual superpotential $\tilde{W}_1(\xi)$ is a real function of ξ. Then equation (43) is an ordinary Shrödinger equation of SUSY quantum mechanics. Using (39) and (44) the solutions of equation (38) can be expressed via the solutions of equation (43) in the following form

$$\phi(x) = \tilde{f}(\xi)\exp\left(\int \mathrm{d}\xi\ \tilde{W}_1(\xi)\right) = \frac{\tilde{f}(\mathrm{i}x)}{\tilde{f}_0(\mathrm{i}x)} \tag{47}$$

where $\tilde{f}(\xi)$ is the solution of equation (43) for the energy ϵ, $f_0(\xi)$ is the solution of the same equation for the zero energy $\epsilon = 0$. Note, that to obtain $\phi(x)$ it is not necessary to use only the square integrable solutions of equation (43). The solutions must be such that $\phi(x)$ is a monotonic function that has one node.

Thus, now we have the problem of solving equation (43). In order to solve this equation the following fact is important. If $W_1(x)$ is such that the corresponding SUSY partners $V_\pm^{(1)}(x)$ belong to the class of shape invariant potentials then $\tilde{W}_1(\xi)$ also gives the shape invariant SUSY partners $\tilde{V}_\pm^{(1)}(\xi)$. To see this recall that the superpotential in the shape invariant case satisfies the following equation [26]

$$W_1^2(x,\alpha) + \frac{\mathrm{d}W_1(x,\alpha)}{\mathrm{d}x} = W_1^2(x,\alpha_1) - \frac{\mathrm{d}W_1(x,\alpha_1)}{\mathrm{d}x} + 2R \tag{48}$$

where the superpotential $W_1(x)$ in the left- and right-hand sides of this equation have different values of parameters α and α_1, the remainder R does not depend on x. From this equation using the definition (46) we obtain

$$\tilde{W}_1^2(\xi,\alpha_1) + \frac{\mathrm{d}\tilde{W}_1(\xi,\alpha_1)}{\mathrm{d}\xi} = \tilde{W}_1^2(\xi,\alpha) - \frac{\mathrm{d}\tilde{W}_1(\xi,\alpha)}{\mathrm{d}\xi} + 2R. \tag{49}$$

As we can see $\tilde{W}_1(\xi)$ also satisfies the shape invariant equation. Note, that in comparison to (48) the set of parameters α in equation (49) is replaced by α_1 and vice versa.

Thus, $\tilde{W}_1(\xi)$ gives the shape invariant SUSY partners $\tilde{V}_\pm^{(1)}(\xi)$ and equation (43) can be solved exactly. Using these solutions on the basis of (47) we get $\phi(x)$. Substituting $\phi(x)$ into (37) we obtain a new exactly solvable potential $V_-(x)$ which is the lower SUSY partner of the known exactly solvable potential $V_+(x)$. Of course we must verify that $\phi(x)$ satisfies the above considered conditions, namely, $\phi(x)$ must be a monotonic function with one node.

In conclusion of this section let us consider some examples.

4.1. Example 3

Let us put

$$W_1(x) = x. \tag{50}$$

Such a choice corresponds to the linear harmonic oscillator. The dual superpotential in this case has the same form as $W_1(x)$

$$\tilde{W}_1(\xi) = \xi \tag{51}$$

and thus equation (43) is the Shrödinger equation for the linear harmonic oscillator. Then, using the well known wavefunctions of stationary states of linear harmonic oscillators on the basis of (47), we obtain $\phi(x) = \beta H_{2k+1}(\mathrm{i}x)$ (β is some constant), which is exactly equal to (22). Here each k corresponds to the energy $\epsilon = 2k + 1$. Note, that in order to satisfy the appropriate conditions for $\phi(x)$ we select only odd solutions of equation (43).

This example reproduces the result of the first example of section 3 in the special case $\gamma = 1$.

4.2. Example 4

Let us consider a superpotential $W_1(x)$ that corresponds to the Rosen–Morse oscillator. In this case

$$W_1(x) = \alpha \tanh(x) \tag{52}$$

is shape invariant. The dual superpotential

$$\tilde{W}_1(\xi) = \alpha \tan(\xi) \tag{53}$$

is also shape invariant. The potential energy corresponding to (53) reads

$$\tilde{V}_-^{(1)}(\xi) = \frac{\alpha(\alpha - 1)}{2\cos^2(\xi)} - \frac{\alpha^2}{2}. \tag{54}$$

As we see the dual potential energy has singularities at the points

$$\xi_n = \frac{\pi}{2} + \pi n \qquad n = 0, \pm 1, \pm 2, \ldots. \tag{55}$$

Traditionally, the Schrödinger equation (43) with potential energy (54) is considered on the interval between two neighbouring singularities using zero boundary conditions for solutions $\tilde{f}(\xi)$. Without any problem we consider the solutions of (43) on the full ξ-line which take zero values in all points (55)

$$\tilde{f}(\xi_n) = 0 \qquad n = 0, \pm 1, \pm 2, \ldots. \tag{56}$$

Such solutions can be easily obtained with the help of SUSY quantum mechanics. Using three first odd solutions of (43) for $\phi(x)$ given by (47) we obtain

$$\phi_1(x) = \sinh(x) \qquad \text{for} \quad \epsilon = \epsilon_1 \tag{57}$$

$$\phi_3(x) = [1 - \alpha + (2 + \alpha)\cosh(2x)] \sinh x \qquad \text{for} \quad \epsilon = \epsilon_3 \tag{58}$$

$$\begin{aligned}\phi_5(x) = {}& [6 + \alpha + 3\alpha^2 - 4(\alpha^2 + 2\alpha - 3)\cosh(2x) \\ & + (\alpha^2 + 7\alpha + 12)\cosh(4x)] \sinh x \qquad \text{for} \quad \epsilon = \epsilon_5\end{aligned} \tag{59}$$

where

$$\epsilon_k = ((\alpha + k)^2 - \alpha^2)/2. \tag{60}$$

Note that we may directly verify that functions $\phi(x)$ given by (57)–(59) indeed satisfy equation (38). Substituting the obtained $\phi_k(x)$ and $\epsilon = \epsilon_k$ into (37) we get the set of exactly solvable potentials $V_-(x,k)$ (here we explicitly write down the dependence of the potential on k)

$$V_-(x,k) = \tanh^2(x)\left(\frac{1}{2}\alpha(\alpha-1) + \Phi_k(x)(\Phi_k(x)+2\alpha)\right) - \epsilon_k + \frac{\alpha}{2} \tag{61}$$

where

$$\Phi_1(x) = 1$$
$$\Phi_3(x) = \frac{3(2+\alpha)\cosh(2x)+\alpha+3}{(2+\alpha)\cosh(2x)-\alpha-1}$$
$$\Phi_5(x) = \frac{(3+\alpha)[5(4+\alpha)\cosh(4x)+4(5-\alpha)\cosh(2x)]+\alpha(5-\alpha)+30}{(3+\alpha)[(4+\alpha)\cosh(4x)-4(1+\alpha)\cosh(2x)]+3(1+\alpha)(2+\alpha)}.$$

The potential $V_-(x,1)$ reproduces the Rosen–Morse one. Other potentials are new exactly solvable ones.

Note that for $k=3$ the potential can be written in the following explicit form:

$$V_-(x,3) = -\frac{4(3+2\alpha)}{((2+\alpha)\cosh(2x)-1-\alpha)^2} + \frac{4(1+\alpha)}{(2+\alpha)\cosh(2x)-1-\alpha} - \frac{(1+\alpha)(2+\alpha)}{2\cosh^2 x} + \frac{(3+\alpha)^2}{2} \tag{62}$$

and was previously obtained by us in [24].

Note, that ϵ is the parameter of the superpotential and thus the parameter of the potentials $V_\pm(x)$. As we see $V_-(x)$ is exactly solvable when ϵ is equal to a certain fixed value. It is worth stressing that for given functions $\phi(x)$ (for example (58), (59)) and arbitrary ϵ it is always possible using the results of the previous section to construct QES potentials with two known eigenstates. These QES potentials at certain fixed values of ϵ (60) become exactly solvable (61) and can be treated as CES potentials.

5. Conclusions

We have developed a new SUSY method for constructing QES potentials for which we know in explicit form the energy levels and wavefunctions of the ground and first excited states. From the obtained general expressions for QES potential and wavefunctions of the ground and first excited states the following interesting fact can be derived. The ratio of the wavefunctions of the first excited state and ground state and the distance between corresponding energy levels entirely determine the potential energy.

The method developed for constructing QES potentials with two known eigenstates is extended for generating CES potentials which are exactly solvable at certain fixed values of parameter ϵ. Finally, this new exactly solvable potential is the lower SUSY partner to the known exactly solvable potential. In this sense our method for generating CES potentials is similar to the method proposed in [20, 21] although the realization is different. In addition, our method gives the interesting relation between QES and CES potentials. Namely, when the parameter ϵ of QES potentials is equal to a certain fixed values then QES potentials become exactly solvable and can be treated as CES ones.

Note, that an important moment in our approach for generating CES potentials is the duality transformation which we use to transform equation (38) to a Schrödinger-type equation. In this paper we only consider the superpotential for which the dual one (46) is a real function of a new variable. The case of complex dual superpotentials is more complicated and we plan to consider this case in future. It will provide a possibility to extend the class of CES potentials which can be obtained by the method suggested in this paper.

References

[1] Singh V, Biswas S N and Dutta K 1978 *Phys. Rev.* D **18** 1901
[2] Flessas G P 1979 *Phys. Lett.* A **72** 289
[3] Razavy M 1980 *Am. J. Phys.* **48** 285
Razavy M 1981 *Phys. Lett.* A **82** 7
[4] Khare A 1981 *Phys. Lett.* A **83** 237
[5] Turbiner A V and Ushveridze A G 1987 *Phys. Lett.* A **126** 181
[6] Turbiner A V 1988 *Commun. Math. Phys.* **118** 467
[7] Shifman M A 1989 *Int. J. Mod. Phys.* A **4** 2897
[8] Zaslavskii O B, Ul'yanov V V and Tsukernik V M 1983 *Fiz. Nizk. Temp.* **9** 511
[9] Zaslavsky O B and Ul'yanov V V 1984 *Zh. Eksp. Teor. Fiz.* **87** 1724
[10] Jatkar D P, Nagaraja Kumar C and Khare A 1989 *Phys. Lett.* A **142** 200
[11] Roy P and Varshni Y P 1991 *Mod. Phys. Lett.* A **6** 1257
[12] Gangopadhyaya A, Khare A and Sukhatme U P 1995 *Phys. Lett.* A **208** 261
[13] Ul'yanov V V, Zaslavskii O B and Vasilevskaya J V 1997 *Fiz. Nizk. Temp.* **23** 110
[14] Ushveridze A G 1994 *Quasi-Exactly Solvable Models in Quantum Mechanics* (Bristol: IOP)
[15] Krajewska A, Ushveridze A and Walczak Z 1997 *Mod. Phys. Lett.* A **12** 1225
[16] Khare A and Mandal B P 1998 *J. Math. Phys.* **39** 3476
[17] de Souza Dutra A 1993 *Phys. Rev.* A **47** R2435
[18] Nag N, Roychoudhury R and Varchni Y P 1994 *Phys. Rev.* A **49** 5098
[19] Dutt R, Khare A and Varchni Y P 1995 *J. Phys. A: Math. Gen.* **28** L107
[20] Junker G and Roy P 1997 *Phys. Lett.* A **232** 155
[21] Junker G and Roy P 1998 *Ann. Phys., NY* **270** 155
[22] Cooper F, Khare A and Sukhatme U 1995 *Phys. Rep.* **251** 267
[23] Junker G 1996 *Supersymmetric Methods in Quantum and Statistical Physics* (Berlin: Springer)
[24] Tkachuk V M 1998 *Phys. Lett.* A **245** 177
[25] Tkachuk V M and Kuliy T V 1998 *Preprint* quant-ph/9807025
[26] Gendenshteyn L E 1983 *Pisma Zh. Eksp. Teor. Fiz.* **38** 299
[27] Shabat A 1992 *Inverse Problems* **8** 303
[28] Spiridonov V P 1992 *Phys. Rev. Lett.* **69** 398
[29] Barclay D T, Dutt R, Gangopadhyaya A, Khare A, Pagnamenta A and Sukhatme U 1993 *Phys. Rev.* A **48** 2786
[30] Bagrov V G and Samsonov B F 1995 *Teor. Mat. Fiz.* **104** 356
[31] Bagrov V G and Samsonov B F 1996 *J. Phys. A: Math. Gen.* **29** 1011